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Advantages of Topological Tools in Localization Methods

Mohammed Khachan and Patrick Chenin

Abstract. Let $C = \{X \in \mathbb{R}^n \ / \ f(X) = 0\}$, $n \in \{2,3\}$, where f is a polynomial function. We want to approximate C by subdividing the parameter space. Most of the usual algorithms raise two problems: data structure management, and the choice of subdivision level which respects the geometry of C. This paper gives a method based on a topological approach. In this work, we specify the local criteria that preserve the topological coherence between the model (the set C) and its volumetric approximation (the set of voxels that contains C). In addition, we determine the local criteria that give the digital analog of $(n-1)$ dimensional manifolds in $\mathbb{R}^n$. In this way, we determine locally how the set of voxels in digital space may be spread out to describe analogous properties of Euclidean manifolds. This gives efficient criteria for controlling the distribution of voxels and the depth of subdivision. We then obtain an approximation that conserves the topological properties of C. The process of localization based on these criteria is generated by an iterative mesh subdivision and skeleton.

§1. Introduction

In recent years, there has been a growing interest in using implicit surfaces for geometric modelling. Especially, the problem of constructing a polygonal approximation of implicit surfaces has received a great deal of attention. The basic idea of all methods for creating a polyhedral approximation of an object is an appropriate subdivision of the relevant space. Polygonalization algorithms typically query the implicit surface through spatial sampling. No preliminary information about the topology of the object is required, and only characterisation coordinates of points are used in the reconstruction. An early polygonalization algorithm for implicit surfaces is described in [2]. It samples the equation of the implicit surface over a three-dimensional rectangular grid of points and linearly interpolates polygons in regions where the function values change signs.

Hall-Warren in [4] and Dessarce-Chenin in [3] presented algorithms based on space subdivision in conjunction with a Bernstein-Bézier representation.

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The Bézier representation is used to exclude regions which cannot contain parts of a surface. Therefore, the algorithms are able to detect also small components which may be not detected by a simple sampling of the defining polynomial on grid points. Furthermore, due to the sampling, the class of algorithms may create new components and merge components.

We say that a polygonalization algorithm does not preserve topology if connected components are not preserved. Finally, we have to propose a nice definition for topological invariance.

In this paper we develop a topological approach related to digital topology theory. It is based on sampling in conjunction with the Bézier representation and a thinning process. The mesh evolution is controlled locally from topological criteria. As far as we know, no author has explored the use of digital topology to control the mesh subdivision for approximating an iso-surface.

The basic idea of our method can be expressed as follows: determination of local criteria for which the voxel set that localizes a given surface has the same geometry as this surface with topology preservation. The geometry is related to manifold properties. For a given subdivision level, if the set of voxels that localize the surface verifies these criteria, we say that the subdivision level reflects the geometric properties of the surface and it's over. Otherwise, we adopt an iterative process coupling two phases (subdivision phase and thinning phase) until the set of voxels represents a digital surface or the upper bound of the subdivision level is reached.

In Section 2, we develop our motivation to use digital topology in localization methods. Section 3 provides some useful definitions and notations related to $3D$-digital topology. In Section 4, we establish the link between Digital and Euclidean topology from the concept of continuous analogous. This enables us to translate properties of polyhedral manifold to digital space. We obtain an efficient and local criterion to determine if a set of voxels is a digital surface. In Section 5, we give criteria for topology preservation and describe the thinning process, provide a brief description of the global algorithm and present some experimental results in $2D$ and $3D$.

§2. Mesh Generation

An implicit surface is given by

$$f(x,y,z) = \sum_{i_k=0}^{n_k} a_{i_1,i_2,i_3} x^{i_1} y^{i_2} z^{i_3} = 0, \text{ where } k \in \{1,2,3\} \text{ and } a_{i_1,i_2,i_3} \in \mathbb{R}.$$

The surface consists of all real points (x, y, z) that verify the above equation. A geometric object is considered as a closed subset of $\mathbb{R}^3$ with the definition $f(x,y,z) \leq 0$, and is called a solid. The boundary of such object is a so-called implicit surface. There is a classification of points in $\mathbb{R}^3$ with respect to the solid. Let $p = (x, y, z)$ a point of $\mathbb{R}^3$. Then

- $f(p) < 0$, *if* p is inside the solid,
- $f(p) = 0$, *if* p is on the boundary of the solid,
- $f(p) > 0$, *if* p is outside the solid.

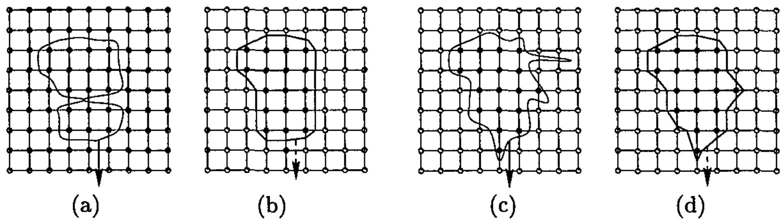

Fig. 1. The topology or the geometry of the initial surface is not preserved.

The classical approach consists of sampling the surface over the three dimensional rectangular grid, called voxels. The decision of whether a voxel is intersected by the surface is made by looking at the function values $f(p)$ at the eight vertices of the voxel. The surface intersects the voxel if not all signs of these values are equal. Thus, the combination of voxels intersecting the boundary of the solid provides an approximation of the whole original surface. Due to the sampling, this approximation cannot always have the same topology as the original surface (see Figure 1 (a,b)). The sign grid points criterion may miss information (see Figure 1 (c,d)): a voxel with all vertices of the same sign can intersect the original surface, but the criterion excludes this voxel class. Hence in order to preserve information, the criterion must keep all voxels that intersect the surface. The Bernstein-Bézier representation allows us to reach this objective.

First, we describe the implicit surface in the Bernstein-Bézier basis. Let $V = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ be a voxel of the space subdivision. The polynomial $f(x,y,z) = \sum_{i_k=0}^{n_k} a_{i_1,i_2,i_3} x^{i_1}.y^{i_2}.z^{i_3}$ with $(x,y,z) \in V$, can be written in Bernstein-Bézier basis form as

$$f(x,y,z) = \sum_{i_k=0}^{n_k} b_{i_1,i_2,i_3}\ B_{i_1}^{n_1}(u).B_{i_2}^{n_2}(v).B_{i_3}^{n_3}(w),$$

with $(u,v,w) \in [0,1]^3$, and $B_l^n(x) = C_l^n.(1-x)^{n-l}.x^l$. The coefficients $b_{i,j,k}$ in $\mathbb{R}$ are the Bernstein-Bézier ordinates. There is a unique set of Bernstein-Bézier ordinates associated with each voxel V; we denote it by $P.C(V)$.

Initially, we have one voxel containing the iso-surface. When we subdivide the voxel in the three directions (x,y,z), the associated Bernstein-Bézier ordinates set is also subdivided (de-Casteljau subdivision) according to the three directions and we relate each new control-polygon with its associate voxel.

Let V be a voxel of the space subdivision. If all the elements of $P.C(V)$ have the same sign, then by the convex-hull property of the Bernstein-Bézier polynomial, f has the same sign over the entire voxel V. Three types of voxels can be found:

1) Outside voxels with $\forall p \in P.C(V),\ p > 0$.
2) Boundary voxels with $\exists p, q \in P.C(V),\ p.q \leq 0$.
3) Inside voxels with $\forall p \in P.C(V),\ p < 0$.

From this partition, we can generate two classes of voxels with respect to the solid, as follows:

- 1-voxel, which corresponds to a boundary voxel,
- 0-voxel, which corresponds to outside or inside voxel.

The convex-hull property of the Bernstein-Bézier polynomial implies that a 0-voxel cannot intersect the surface, but does not assert that a 1-voxel intersects the surface. So, the approximation cannot always have the same topology as the original surface (we can create components that do not exist in the original surface and merge components).

In order to overcome this problem, we develop a topological approach related to digital topology theory. It consists of representing the space subdivision by a binary three dimensional digital image. The 0-voxel represents a voxel of the background, and the 1-voxel a voxel of the image object. In order to avoid having to consider the boundary of the 3-digital image, we assume that the digital image is unbounded in all directions.

Let λ be a centroid-map which associates to each voxel σ its barycentre $\lambda(\sigma)$. λ is an one-to-one map between the set of all voxels in $\mathbb{R}^3$ and its associate digital grid $\mathbb{Z}^3$. Points of $\mathbb{Z}^3$ associated with 1-voxels are called black points, and those associated with 0-voxel are called white points. The set of black points normally corresponds to an object in the digital grid.

In the next section we recall the definition of a binary three-dimensional digital grid.

§3. Basic Notions in 3D-digital Grids

A significant concept in the study of a digital grid is that of neighborhood. By means of the neighborhood we are able to define "topology" in the digital space. A point $p \in \mathbb{Z}^3$ is defined by $(x_i(p))_{i=1}^3$ with $x_i(p) \in \mathbb{Z}$. We consider two types of neighbors of p in the $3D$-Digital Grids

- The 6-neighbors: $\mathcal{N}_6(p) = \{q \in \mathbb{Z}^3 \ : \ \sum_{i=1}^{3} |x_i(p) - x_i(q)| = 1\}$,
- The 26-neighbors: $\mathcal{N}_{26}(p) = \{q \in \mathbb{Z}^3 \ : \ \max_{1 \le i \le 3} \left(|x_i(p) - x_i(q)|\right) = 1\}$.

Let $\beta \in \{6, 26\}$ and $T \subset \mathbb{Z}^3$. We say that q is β-adjacent to p *if and only if* $q \in \mathcal{N}_\beta(p)$. p is said to be β-adjacent to T if p is β-adjacent to some point in T. Two sets T and W are said to be β-adjacent to each other if some point in T is β-adjacent to some point in W.

A β-path of length m, $m > 0$, from p to q in T means a sequence of distinct points $p = p_0, \dots, p_m = q$ of T such that p_i is β-adjacent to p_{i+1}, $1 \le i < m$. Two points $p, q \in T$ are β-connected in T if and only if there exists a β-path from p to q in T. The equivalence classes of T under β-connectivity are called β-components of T. A set of points T is called β-connected if and only if every two points p, q in T are β-connected in T.

- $\mathcal{N}_T(p)$ denotes the set $\mathcal{N}(p) \cap T$.
- $\mathcal{N}_{\beta,T}(p)$ denotes the set of elements in T that are β-adjacents to p,

- T^c denotes the complement of T in $\mathbb{Z}^3$; $T^c = \mathbb{Z}^3 - T$.

We let $\mathcal{N}(p)$ denote the 27 points in the $(3,3,3)$ neighborhood of p. A point p in T such that $\mathcal{N}(p) \subset T$ is called an `interior` point, otherwise p is called a border point [10].

In this paper, our neighborhood structure corresponds to the $(6,26)$-adjacency relation: 6-adjacency for the object and (26)-adjacency for its complement in the digital image.

§4. Continuous Analogs and 2-digital Manifold in $\mathbb{Z}^3$

The notion of continuous analogs was introduced in [9] to establish general properties of binary three dimensional images and used in [8] to give a natural proof of a theorem on simple surface points.

Generally, this tool permits us to relate digital topology to polyhedral topology. In [7] we generalize the concept of continuous analogs in all dimension n, in the context of $(2n, 3^n - 1)$-adjacency, and establish the link between digital and Euclidean topology.

In the following, we recall some results in three dimension space given in [7] for all dimension space. Let T be a subset of $\mathbb{Z}^3$. We will construct a polyhedral complex $C(T)$ of T as follows:

- A `0-cell` of $C(T)$ is an element of T,
- A `1-cell` of $C(T)$ is an unit segment whose vertices belong to T,
- A `2-cell` of $C(T)$ is an unit square whose vertices belong to T,
- A `3-cell` of $C(T)$ is an unit cube whose vertices belong to T.

$C(T)$ is a complex, it is called the `cubical-complex` of T. The underlying space $|C(T)|$ is called the `continuous analog` of T in $\mathbb{R}^3$. Note that [5] gives a general method for generating polyhedra from a set of lattice points in $\mathbb{Z}^3$. The following theorem expresses the fundamental properties of continuous analogs. The proof is given in [7].

Theorem 1. *Let T be a subset of $\mathbb{Z}^3$.*

1) $|C(T)| \cap \mathbb{Z}^3 = T$,
2) *Two elements of T are in the same 6-component of T if and only if they are in the same component of $|C(T)|$,*
3) *Two elements of T^c are in the same 26-component of T^c if and only if they are in the same component of $\mathbb{R}^3 - |C(T)|$.*

The remainder of this section deals with the relation between a $2D$-digital surface, a $2D$-polyhedral manifold and the Jordan-Brouwer Theorem. In $3D$ Euclidean space, a simple closed surface is well defined: the neighborhood of each point in the surface is homeomorphic to an Euclidean disc. The analog property in digital space consists of caracterizing a 'surface' in $\mathbb{Z}^3$ by considering its associate continuous analog set.

Morgenthaler and Rosenfeld in [12] have introduced the notion of simple closed digital surface in order to establish a nontrivial $3D$-analog of the $2D$-Jordan curve theorem. They characterized a simple closed digital surface as a connected collection of *orientable* simple surface points. In [8] T.Y. Kong and A.W. Roscoe reveal what simple surface points 'look like'. In [6], we establish in all dimensions the relation between n-dimensional digital manifold and n-dimensional Euclidean manifold. Here, we give only properties related to the caracterisation of a digital surface. General results and proof are given in [6].

Definition 2. *Let T be a subset of $\mathbb{Z}^3$ and $p \in T$. p is called* a simple surface point *of T if*

- *$\mathcal{N}_{T^c}(p)$ admits exactly two 26-components, denoted by $Int(p)$ and $Ext(p)$,*
- *$\forall q \in \mathcal{N}_{6,T}(p)$, q is 6-adjacent to $Int(p)$ and $Ext(p)$.*

Definition 3. *Let T be a subset of $\mathbb{Z}^3$ and $p \in T$. T is called* a digital surface if *$p \in T$, p is a simple surface point of T.*

Theorem 4. *Let T be a subset of $\mathbb{Z}^3$.*

- *T is a digital surface if and only if $|C(T)|$ is a simple and closed surface in $\mathbb{R}^3$.*
- *if T is a digital surface, then T^c has exactly two 26-components ($Int(T)$ and $Ext(T)$), and every element of T is 26-adjacent to these components.*

Let T be the set of black points in the digital grid. The notion of simple surface point gives an efficient and local criterion to determine if T is a digital surface. This criterion requires only a small number of local operation per point of T. By translating this characterisation in the voxel space, we obtain an efficient criterion to determine if a set of voxels represent a digital surface.

§5. Algorithm and Results

For a given subdivision level, the above criterion allows us to test if the set of 1-voxels represents a digital surface. If it is true, we say that the current subdivision level reflects the geometric properties of the original surface and it's over, otherwise our approach consists of two phases.

During the first phase, we subdivide the set of 1-voxels in the three directions and label the new voxels (1-voxel and 0-voxel). During the second phase, we use a thinning process (remove 0-voxels with topology preservation). We adopt a sequential thinning process: the border voxels with 0 values are 'peeled off' layer-by-layer with topology preservation. The remaining digital set, called the skeleton, contains all 1-voxels of the current level and will contain 0-voxels whose deletion would destroy the current topology.

The second phase permits us to control the topology evolution. We can merge components of the skeleton's complementary set or create a hole in the digital image, by removing a specific 0-voxel (if there exists) from the skeleton. The end step of this phase is to label all the voxels of the skelton to 1 (1-voxel).

Topological thinning is a widely used approach for generating skeletons from binary objects. It has been shown (see [10,1]) that the topology in a digital grid will be preserved by a thinning process if the border points that are removed during each step are simple points. The notion of simple point is related to topology preservation. A border point is a simple point *if and* *only if* the Euler number and the number of connected components in its neighborhood does not change after its removal [10,6,11].

The body of our method, called the treatment phase, is organised in an iterative way, each step consists of the two phases described above, until the skeleton represents a digital surface or the upper bound of the subdivision level is reached.

Input

- Equation of the Implicit Surface, $f(x, y, z)$,
- Cuboid containing the surface.

Ouput

- A set of voxels that localizes the surface with topology and geometry preservation.

We begin by tranforming f in the Bernstein-Bézier basis related to the Cuboid. Our algorithm consists of two phases: initialization phase and treatment phase.

> During the initialization phase, we extract the components of the object and its complement (Interior and Exterior components).
>
> During the treatment phase, we apply iteratively the thinning and subdivision process until the skeleton represents a digital surface or the upper bound of the subdivision level is reached.

The subdivision process consists of combining the voxel space subdivision with control polygon subdivision. The thinning phase proceeds in an iterative way

- Update the border component of the object,
- For each border component, remove sequentially 0-voxels that correspond to a simple point.

We note that this work remains valid for any notion of digital surface satisfying the Jordan-Brouwer theorem.

5.1 Illustration in 2D case

We consider the following function written in terms of the Bernstein-Bézier basis:

$$\begin{aligned} f(x,y) &= 4700 - 40670x - 5000y + 160965x^2 + 5000y^2 - 264750x^3 + 155584x^4 \\ &= \sum_{i=0}^{2} \sum_{j=0}^{4} p_{ij} B_i^4(x) B_j^4(y). \end{aligned}$$

This example illustrates the concept of our approach. If the surface has no-singularity, our method gives the subdivision level for which the set of 1-voxels corresponds to a digital surface. The following example illustrates the

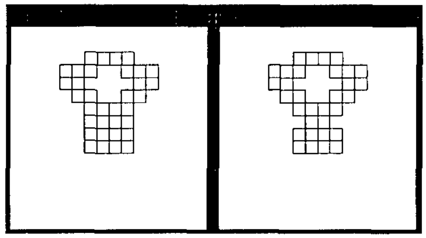

Fig. 2. Initialization phase.

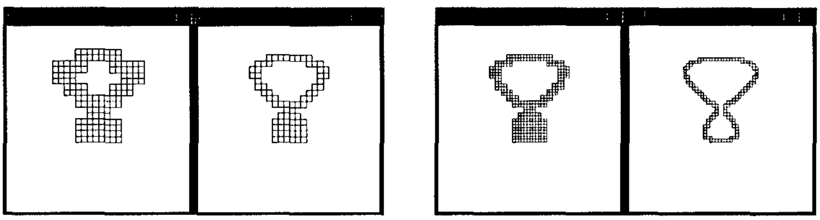

Fig. 3. Treatment phase.

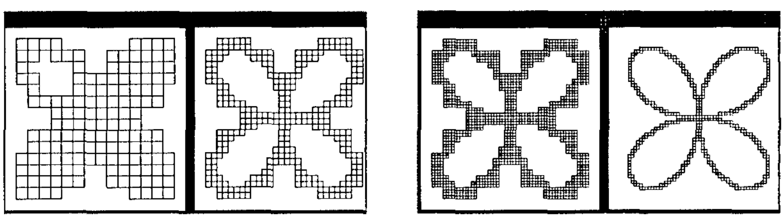

Fig. 4. Initialization and first step of treatment phase.

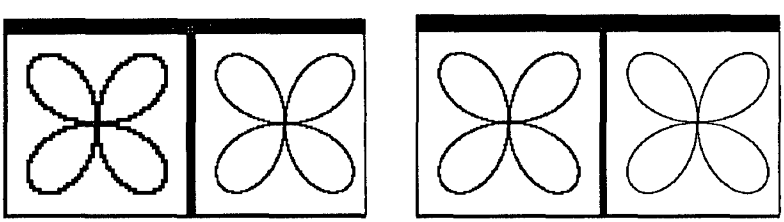

Fig. 5. Final result.

case where the initial object admits a singularity. Let

$$\begin{aligned} f(x,y) = 4 &- 32x + 128x^2 - 256x^3 + 288x^4 - 192x^5 + 64x^6 \\ &+ 128xy - 320x^2y - 320xy^2 + 384x^3y + 512x^2y^2 + 384xy^3 \\ &- 192x^4y - 384x^3y^2 - 384x^2y^3 - 192xy^5 - 32y \\ &+ 128y^2 - 256y^3 + 288y^4 - 192y^5 + 64y^6. \end{aligned}$$

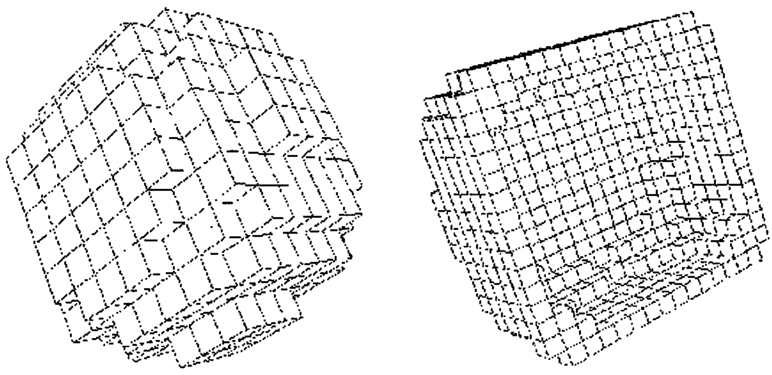

Fig. 6. A digital sphere.

5.2 Illustration in 3D case

Figure 6 illustrates the algorithm's application to a sphere given by an implicit equation. It shows external and internal sights.

§6. Conclusion

In this paper we use a digital topology approach to preserve the topological coherence between the model (original surface) and its volumetric approximation. Our method is based on subdivision and a thinning process. We use local and efficient criteria to determine the nature of the approximation and to preserve the current topology.

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